Problem 32) a) $f(x) = [\text{Rect}(x)\cos(\pi x)] * \text{Comb}(x)$.

b) First, we find the Fourier transform of one period of the function $|\cos(\pi x)|$ in the interval $[-½, +½]$, as follows:

$$\int_{-½}^{½} \cos(\pi x)\exp(-\mathrm{i}2\pi sx)\,dx$$

$$= \int_{-½}^{½} ½[\exp(\mathrm{i}\pi x) + \exp(-\mathrm{i}\pi x)]\exp(-\mathrm{i}2\pi sx)\,dx$$

$$= ½\int_{-½}^{½} \exp[\mathrm{i}\pi(1-2s)x]\,dx + ½\int_{-½}^{½} \exp[-\mathrm{i}\pi(1+2s)x]\,dx$$

$$= \frac{\exp[\mathrm{i}\pi(½-s)] - \exp[-\mathrm{i}\pi(½-s)]}{\mathrm{i}2\pi(1-2s)} - \frac{\exp[-\mathrm{i}\pi(½+s)] - \exp[\mathrm{i}\pi(½+s)]}{\mathrm{i}2\pi(1+2s)}$$

$$= \frac{2\mathrm{i}\cos(\pi s)}{\mathrm{i}2\pi(1-2s)} + \frac{2\mathrm{i}\cos(\pi s)}{\mathrm{i}2\pi(1+2s)} = \frac{2\cos(\pi s)}{\pi(1-4s^2)}. \quad (1)$$

Alternatively, invoking the fact that a single period of $f(x)$ is represented by $\text{Rect}(x)\cos(\pi x)$, we may use the convolution theorem to write

$$\mathcal{F}\{\text{Rect}(x)\cos(\pi x)\} = \text{sinc}(s) * ½[\delta(s+½) + \delta(s-½)]$$

$$= \frac{\sin[\pi(s+½)]}{2\pi(s+½)} + \frac{\sin[\pi(s-½)]}{2\pi(s-½)} = \frac{\cos(\pi s)}{\pi(2s+1)} - \frac{\cos(\pi s)}{\pi(2s-1)} = \frac{2\cos(\pi s)}{\pi(1-4s^2)}. \quad (2)$$

Considering that the convolution of one period of $f(x)$ with $\text{comb}(x)$ produces the periodic function $|\cos(\pi x)|$, the Fourier transform of $f(x)$ must be the product of $\text{comb}(s)$ and the Fourier transform of a single period of $f(x)$, which is given by Eqs.(1) and (2). The Fourier series coefficients of $f(x)$ are thus given by

$$F_n = \frac{2\cos(n\pi)}{\pi(1-4n^2)} = \frac{(-1)^n}{(1-4n^2)(\pi/2)}. \quad (3)$$

The Fourier series representation of $f(x)$ is readily found to be

$$f(x) = |\cos(\pi x)| = \sum_{n=-\infty}^{\infty} F_n \exp(\mathrm{i}2\pi nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2\pi nx)}{4n^2-1}. \quad (4)$$